Non-linear dynamics of a two phase flow system in an evaporator: The effects of (i) a time varying pressure drop (ii) an axially varying heat flux

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Abstract

In this paper we study the phenomena of density wave oscillations (DWO) in a vertical heated channel. The homogeneous equilibrium model is used to simulate the flow in the two-phase region. The equations are solved numerically using a ‘shooting-method’ technique. This in its turn employs an implicit backward finite difference scheme. The scheme can incorporate the movement of the interface. It is very elegant and does not involve storage of variables in large $N \times N$ matrices. This scheme is sufficiently general and can be used to simulate the dynamic behaviour when: (i) the heat flux imposed at the surface is non-constant, i.e. exhibits an axial variation; and (ii) the imposed pressure drop is varied periodically at a fixed frequency. A possible explanation for the conflicting reports of the effect of a periodic variation in heat flux is provided using a linear stability analysis and the D-partition method. The interaction of the natural frequency of the DWO and the fixed forcing frequency of the imposed pressure drop gives rise to various phenomena viz relaxation oscillations, sub-harmonic oscillations, quasi-periodic and chaotic solutions. To aid the experimentalist describe this infinite-dimensional system on the basis of his experimental results we discuss the characterisation using only the velocity time series data. This is done employing the method of delay coordinate embedding. The phase portraits, stroboscopic map and correlation dimension of the actual attractor are compared with that of the reconstructed attractor from the velocity time series. © 1997 Elsevier Science S.A.

1. Introduction

Density wave oscillations (DWO) is a frequently encountered dynamic instability in two phase (liquid-gas) flow systems. This is exhibited typically by evaporating systems. It can also occur in the cooling coil of a nuclear reactor where the heat flux can be sufficiently high to vapourise the coolant.

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velocity (when the instability is dynamic). This oscillatory behaviour is self-sustained and is caused by non-linear interactions in the system. These periodic fluctuations can induce cyclic stresses and cause mechanical damage to equipment.

Quandt (1961) was the first to differentiate between the two kinds of instabilities: static and dynamic. He showed that the former occurred in the negative slope region of the steady-state pressure-drop characteristic curve and the latter in the positive slope region. He solved for the dynamic behaviour using the channel integral model. In this approach the transient behaviour is modelled such that the spatial variation of the enthalpy profile and other dependent variables are taken to be the same as those prevailing at steady-state. Stenning and Veziroglu (1963) were the first to physically explain the phenomena of DWO. Saha et al. (1976) studied experimentally the effect of various parameters like inlet sub-cooling, inlet restriction, exit restriction on the stability of the system to DWO. Peng et al. (1982) developed a code called NUFREQ-NP to simulate the interaction between neutron-kinetics and multiphase flow.

The first detailed theoretical study of DWO was carried out by Achard et al. (1985) using the homogeneous equilibrium model (HEM) for the two phase flow. They obtained the marginal stability boundary (MSB) across which DWO occur and the boundary across which Ledinegg instability occurs (Ledinegg stability boundary or LSB) in the two-dimensional parameter space of sub-cooling number and friction number using the D-partition method. They transformed their system of hyperbolic equations to an integro-differential system in inlet velocity using the method of characteristics for their analysis. Rizwan-uddin and Dorning (1986) followed the method of Achard et al. and used the drift-flux model. They obtained the MSB for DWO in the parameter space of sub-cooling number and phase change number and compared their results with experiments. In a later study (Rizwan-uddin and Dorning, 1988, 1990) the effect of a periodic perturbation in the imposed pressure drop on the system behaviour was considered. It was shown that the system can exhibit sub-harmonic or quasi-periodic or chaotic behaviour depending on the amplitude and the frequency of the periodic forcing. A limitation of this approach based on the method of characteristics is that it can be applied only when the heat flux at the surface is independent of axial position.

The numerical codes existing in the literature like PHOENICS, SETS assume a pressure variation in the channel (Jones (1987)). They solve for the velocity variation in the channel and iterate on the pressure profile using the equation of continuity. This technique involves the storage of variables in several large matrices and is computationally intensive since it does not exploit the features of the system.

Most of the theoretical and experimental studies to date have analysed the behaviour of the above system in a vertical channel subject to a uniform heat flux. A schematic representation of this idealised system is shown in Fig. 1.

In this study we have used a ‘shooting method’ technique to simulate the oscillatory behaviour of the evaporating system employing the homogeneous equilibrium model. An implicit backward finite difference scheme is used to discretise the governing partial-differential equations. We discuss the detailed algorithm of the numerical

![Fig. 1. Schematic diagram of system investigated.](image)
scheme which is sufficiently versatile to accommodate an axial variation of the heat flux. The formulation of the numerical scheme is such that it avoids storage of variables in large $N \times N$ matrices thus overcoming one of the major disadvantages of the current existing general purpose numerical codes. The present numerical code is verified by comparing our results with those available in the literature. The code is then used to study the effect of axial variation of heat flux on the stability.

We also study the effect of a periodic variation in the imposed pressure drop across the channel. This introduces an additional time scale into the system. The dynamic behaviour of this system is simulated numerically using our algorithm. Besides identifying the well known subharmonic, quasi-periodic and chaotic solutions we find the occurrence of relaxation oscillations due to the presence of two widely different time scales in the system. We use the delay coordinate embedding technique on the velocity time series to characterise system behaviour quantitatively. The correlation dimension of the reconstructed attractor quantifies the system behaviour. This is validated by comparing the correlation dimension of the reconstructed attractor with the attractor dimension of the original system. This technique is useful for quantifying the behaviour of an infinite dimensional system on the basis of experimental results.

2. Model

The model we use to represent the system behaviour is based on the following assumptions:

1. The flow is assumed to be one-dimensional, i.e. radial variations of all quantities are neglected.
2. The liquid phase and the gas phase are both incompressible. The density in the two phase region varies due to change in the quality or void fraction of the system as we go up the channel.
3. The homogeneous equilibrium model is valid in the two phase region.
4. The two phases are always in thermal equilibrium. There is no sub-cooled boiling and a well defined boundary separates the single-phase (liquid) region from the two-phase (liquid–vapor) region.
5. The pressure drop inside the channel is negligible as compared to the inlet pressure $P_{in}$ for the purposes of calculation of the saturation enthalpy. This facilitates the computation of the interface position, since it decouples the equation of motion from the equation of energy.

The first two assumptions are valid at high velocities, large length to tube diameter ratios and high system pressures, respectively. Assumptions 3 and 4 have been shown to be appropriate for theoretical predictions by Saha and Zuber (1978). These authors have considered a drift flux model along with thermal non-equilibrium and have found the stability boundaries quite close to those predicted by the homogeneous equilibrium model.

The dimensionless equations modelling the system are different in the single phase region and in the two phase region. Denoting the interface position by $\lambda(t)$ we have for the single phase region $0 < \eta < \lambda(t)$

$$\frac{\partial v_{in}}{\partial \eta} = 0$$

$$- \frac{\partial P_{1\phi}}{\partial \eta} = \frac{\partial v_{in}}{\partial \tau} + N_{f1}v_{in}^2 + Fr^{-1}$$

$$\frac{\partial h}{\partial \tau} + v_{in} \frac{\partial h}{\partial \eta} = \frac{N_{\rho}}{1 - N_{\rho}} N_{pch}(\eta)$$

In the two phase region $\lambda(t) < \eta < 1$

$$\frac{\partial j}{\partial \eta} = N_{pch}(\eta)$$

$$\frac{\partial \rho_M}{\partial \tau} + j \frac{\partial \rho_M}{\partial \eta} = - \rho_M N_{pch}(\eta)$$

$$- \frac{\partial P_{2\phi}}{\partial \eta} = \rho_M \left( \frac{\partial j}{\partial \tau} + j \frac{\partial j}{\partial \eta} + Fr^{-1} + N_{j2}j^2 \right)$$

The characteristic variables used in making the variables dimensionless are the channel length $L^*$, the liquid density $\rho^*$, the characteristic velocity $v_0^*$ and the latent heat $\Delta h^*_{le}$. In the above equations, $v_{in}$ represents inlet velocity, $h$ the enthalpy, $P$ the
pressure, \( j \) the volumetric flux in the two phase region and \( \rho_M \) the two-phase mixture density. The dimensionless groups \( N_{f1} \) and \( N_{f2} \) are the friction factors corresponding to the single phase and two-phase regions. \( N_{\text{pch}} \) the phase change number is a dimensionless heat flux incident on the wall, \( Fr \) is the Froude number and \( N_s \) represents the dimensionless density ratio of the two phases. All the variables are defined in the nomenclature section. The superscript ‘*’ denotes dimensional variables. In addition to the pressure drop in the single phase and two-phase regions in the channel, we consider the pressure drop arising from the inlet and exit restrictions.

The dimensionless inlet and exit losses are given by

\[
\Delta P_{\text{in}} = k_{\text{in}} \frac{v_{\text{in}}^6}{N_s^{1/2}} \tag{7}
\]

\[
\Delta P_{\text{ex}} = k_{\text{ex}} \rho_s (\eta = 1, \tau) j(\eta = 1, \tau) \tag{8}
\]

where \( k_{\text{in}} \) and \( k_{\text{ex}} \) are the inlet and exit restriction coefficients. The system is subject to either an externally imposed constant or a varying pressure drop so that at any instant of time.

\[
\Delta P_{\text{tot}}(\tau) = \Delta P_{\text{imp}}(\tau)
= \Delta P_{\text{in}}(\tau) + \Delta P_{\text{ex}}(\tau) + \Delta P_{\text{i} \phi}(\tau) + \Delta P_{\text{2} \phi}(\tau)
\tag{9}
\]

3. Method of solution

3.1. Steady-state solution

The steady-state of the system can be obtained by setting the partial derivatives with respect to time as zero in Eqs. (1)–(6). The resulting system of first order ordinary differential equations in space are integrated as an initial value problem for a fixed \( v_{\text{in}} \) to obtain the steady-state pressure drop characteristics. These represent the dependence of pressure drop on steady-state inlet velocity, \( v_{\text{in}}^{ss} \).

For a vertical channel (with finite \( Fr \)), the individual pressure drops are given by

\[
\Delta P_{\text{in}}^{ss} = k_{\text{in}} v_{\text{in}}^{ss2}
\]

\[
\Delta P_{\text{ex}}^{ss} = N_{f1} v_{\text{in}}^{ss2} \lambda^{ss} + Fr^{-1} \lambda^{ss}
\]

\[
\Delta P_{\text{i} \phi}^{ss} = \frac{v_{\text{in}}^{ss}}{Fr N_{\text{pch}}} \ln \frac{v_{\text{in}}^{ss} + N_{\text{pch}}(1 - \lambda^{ss})}{v_{\text{in}}^{ss}}
+ \frac{N_{\text{pch}}(1 - \lambda^{ss})^2 N_{f2} v_{\text{in}}^{ss}}{2} + N_{\text{pch}}(1 - \lambda^{ss}) v_{\text{in}}^{ss}
\]

\[
\Delta P_{\text{2} \phi}^{ss} = k_{\text{ex}} v_{\text{in}}^{ss} j_{\text{ex}}^{ss}
\]

where \( j_{\text{ex}}^{ss} \) is the steady-state exit velocity of the two phase mixture given by,

\[
j_{\text{ex}}^{ss} = \frac{v_{\text{in}}^{ss} + N_{\text{pch}}(1 - \lambda)}{Fr}
\]

The total pressure drop is given by the sum of these individual pressure drops as long as \( \lambda < 1 \) or \( v_{\text{in}}^{ss} < N_{\text{pch}}/N_{\text{sub}} \). For \( v_{\text{in}}^{ss} \) larger than these values the channel is completely filled by the liquid and the pressure drop is given as

\[
\Delta P_{\text{tot}}^{ss} = k_{\text{in}} v_{\text{in}}^{ss2} + N_{f1} v_{\text{in}}^{ss2} + Fr^{-1} + k_{\text{ex}} v_{\text{in}}^{ss2}
\]

The dependence of steady-state pressure drop on inlet velocity is shown in Fig. 2, for two different parameter sets. For set 1, \( (k_{\text{in}}, k_{\text{ex}}, Fr^{-1}, N_{f1}, N_{f2}, N_{\text{sub}}, N_{\text{pch}}) = (6, 2, 30, 2.8, 5.6, 7.5, 13) \) there is a range of pressure drops for which the system exhibits multiple steady-states and for set 2 we have a unique steady-state for all pressure drops. In the region of multiplicity, the intermediate steady-state (in the negative slope region) is unstable. Deviations from this state force the system to evolve to one of the other possible states (i.e. those along the positive slope branches). The operating points in the negative slope region are statically unstable.

4. Dynamic simulation

The time dependent behaviour of the system can be obtained by solving the coupled system of hyperbolic partial differential Eqs. (1)–(6). The numerical technique used for simulations can be rendered more efficient by analytical preprocessing, which results in some simplifications.

4.1. Analytical preprocessing

Eq. (2) for example can be integrated analytically to obtain the pressure drop in the single phase region at every time instant as
Integrating Eq. (4) as an initial value problem subject to the conditions in Eq. (11) yields,

\[ j = v_{\text{in}}(\tau) + N_{\text{pch}}(\eta - \hat{\lambda}(\tau)) \]  

(12)

So the time dependency of the flux \( j \) is induced by the variations of the inlet velocity \( v_{\text{in}}(\tau) \) through Eq. (11). In our numerical method for solving the equations we exploit these analytical relationships.

4.2. Location of interface

The dynamic simulations are further complicated by the varying extent of the single phase and the two-phase regions. The location of the interface separating these regions can be obtained in two possible ways. In the first the method of characteristics is used (following Rizwan-uddin) to yield

\[ \hat{\lambda}(\tau) = \int_{\tau - \tau_1}^{\tau} v_{\text{in}}(s) ds \]  

(13)

where \( \tau_1 \) is the single phase residence time and is given by

\[ \tau_1 = \frac{N_{\text{sub}}}{N_{\text{pch}}} \]

\( N_{\text{sub}} \) is the subcooling number and it indicates how far the inlet enthalpy is from saturation enthalpy \( h_{\text{sat}} \). The interface position \( \hat{\lambda}(\tau) \) can also be located using the method of finite differences. Assuming \( h_{\text{sat}} \) is independent of time and space we can rewrite Eq. (3) as

\[ \frac{\partial}{\partial \tau} (h(\eta, \tau) - h_{\text{sat}}) + v_{\text{in}}(\tau) \frac{\partial}{\partial \eta} (h(\eta, \tau) - h_{\text{sat}}) \]

\[ = \frac{N_{\rho}}{1 - N_{\rho}} N_{\text{pch}}(\eta) \]

(14)

or

\[ \frac{\partial N_{\text{sub}}(\eta, \tau)}{\partial \tau} + v_{\text{in}}(\tau) \frac{\partial N_{\text{sub}}(\eta, \tau)}{\partial \eta} = -N_{\text{pch}}(\eta) \]

The implicit finite difference form of the above equation at the \( i \)th grid point using a backward difference is:
\[
\frac{N_{\text{sub}}(i, \tau + \Delta \tau) - N_{\text{sub}}(i, \tau)}{\Delta \tau} + v_{\text{in}}(\tau + \Delta \tau) \frac{N_{\text{sub}}(i, \tau + \Delta \tau) - N_{\text{sub}}(i - 1, \tau + \Delta \tau)}{\Delta \eta} = -N_{\text{pch}}(i)
\] (15)

Such a discretization permits us to extend our analysis to problems where the heat flux \(N_{\text{pch}}\) varies with axial position. The location of the boundary, \(\lambda(\tau)\) at any instant of time is given by the grid position where \(N_{\text{sub}} = 0\) from Eq. (15). This is in contrast to Eq. (13) which can be used only when the heat flux is independent of axial position.

The evolution of the system to a dynamic state necessitates solving the coupled system Eqs. (1)–(6) numerically. We have employed an extension of the classical ‘shooting method’ technique used in solving two point boundary-value problems. We iterate for the inlet velocity at every time instant such that the channel pressure drop equals the imposed pressure drop at that instant. For purposes of higher accuracy and stability we use an implicit finite difference method to discretise the equations spatially. The finite-difference in spatial direction has to be a backward difference scheme. This is important as the velocity in the entire channel is positive. Consequently the information in the system flows from bottom to top. This is called the upwind difference scheme (Ozičik (1994)).

We adopted a finite-difference method with a view to generalising present methods of simulating the system dynamics. Our numerical scheme can include the effect of an axial variation in heat flux which enables one to include the effect of neutron-kinetics, nuclear reaction etc. The method of characteristics employed by Achard et al. (1985) can be used only when the heat flux is constant. The single phase residence time and consequently the interface position for example cannot be evaluated in this technique if the heat flux varies with axial position.

Another important feature of the proposed numerical technique is that it can be implemented without employing any matrix operations. This is rendered possible since we solve only for a scalar variable \(v_{\text{in}}(\tau)\). All other variables can now be stored in an array form. This reduces the memory requirements on the computer drastically and makes the computations very efficient. The other general purpose numerical codes which exist store variables in large matrices for their implementation. This is because the strategy adopted is to satisfy the equations of motion assuming a given axial pressure variation. They iterate on the pressure profile (a vectorial quantity) by satisfying the equation of continuity.

4.3. Numerical algorithm

The proposed numerical scheme consists of the following steps.

1. From the steady state equations the steady-state value of the interface position and the steady value of the pressure drop are calculated for a fixed inlet velocity. The steady-state pressure drop is taken to be the imposed pressure drop for the transient analysis.

2. The total length of the channel is divided into ‘\(N\)’ equal grids. While computing the interface for the dynamic simulations using the method of characteristics \(N\) was taken to be 650 and when computing using the finite difference scheme, i.e. Eq. (15) \(N\) was taken as being 1400. These values were arrived at by considering the accuracy of the algorithm in the numerical simulations. No significant improvement in accuracy was obtained by increasing \(N\) beyond these values, respectively.

3. The dynamic simulation is triggered by using a small perturbation of the inlet velocity. The velocity and density at all other grid points at this time instant are taken to be equal to the steady-state value.

4. A value for the velocity at the next time step is assumed. The location for the boiling boundary for this time instant is obtained. Since the calculation of the interface position using the method of characteristics (Talyerkhan et al., 1981) involves velocity histories, the velocity perturbation while using this method is imposed at \(\tau = \tau_{1\phi}\) and not at \(\tau = 0\). The system is assumed to be at steady-state till \(\tau = \tau_{1\phi}\). While using the method of finite-differences to obtain \(\lambda\) we impose the velocity perturbation at \(\tau = \tau_{1\phi}\). The pressure drop in the single phase region is obtained using Eq. (10).
5. The density and velocity at each point in the two-phase region are obtained from the discretised equations. Eq. (5) for example is transformed as

\[
\frac{\rho_M(i, \tau + \Delta \tau) - \rho_M(i, \tau)}{\Delta \tau} + j(i, \tau + \Delta \tau) 
\times \frac{\rho_M(i, \tau + \Delta \tau) - \rho_M(i - 1, \tau + \Delta \tau)}{\Delta \eta} 
= N_{\text{pch}} \rho_M(i, \tau + \Delta \tau)
\]

6. The pressure gradient at each point in the two-phase region is obtained from the discretised form of Eq. (6). This is integrated using a trapezoidal rule to get the two-phase pressure drop. The total pressure drop is obtained by adding the individual pressure drops.

7. The total pressure drop at every time instant is compared with the imposed pressure drop. If they match to within an allowed tolerance the guessed value of \(v_{in}(\tau)\) is assumed to be correct and we proceed to the next time step. Otherwise, a Newton–Raphson scheme is used to obtain a better estimate of the inlet velocity using

\[
v_{\text{in}}^{\text{new}}(\tau + \Delta \tau) = v_{\text{in}}^{\text{old}}(\tau + \Delta \tau) - (\Delta P_{\text{tot}} - \Delta P_{\text{imp}}) \left( \frac{\partial \Delta P_{\text{tot}}}{\partial v_{\text{in}}} \right)^{-1}
\]

The derivative here is calculated numerically. This reduces the method to a regula falsi method. The integration is performed for a sufficiently long time to study the dynamic behaviour. In our simulations we used a tolerance value of \(2 \times 10^{-4}\) in the Newton–Raphson step for a relative error in inlet velocity and a time step of 0.005 for the numerical integration. The numerical derivative was obtained using a 0.5% difference in \(v_{\text{in}}^{\text{old}}\).

5. Results and discussion

We start this section with a discussion on the effect of an axially varying heat flux on the stability of the heated channel. In all the results that we present below we use the parameters belonging to set I of Fig. 2, unless explicitly specified otherwise.

5.1. Axial variation in heat flux

Study of axial heat flux variation is important because the heat flux variation in nuclear reactors is found to follow a cosine variation (Todreas and Kazimi (1990)). For the case of the constant imposed pressure drop we have simulated the effect of an axial variation in \(N_{\text{pch}}\). The spatial variation of \(N_{\text{pch}}\) is assumed to be of the form

\[
N_{\text{pch}}(\eta) = \frac{N_{\text{av}}^{\text{pch}} f(\eta)}{\int_{0}^{1} f(\eta) \, d\eta}
\]

Three different axial variations are considered
- Linear: \(f(\eta) = 1 + \epsilon \eta\)
- Exponential: \(f(\eta) = e^{\epsilon \eta}\)
- Cosine: \(f(\eta) = 1 + \epsilon \cos 2\pi k \eta\)

The parameter \(\epsilon\) characterises the deviation from the uniform heat flux distribution. The parameter \(k\) is a measure of the frequency of the periodic distribution. The parameter \(N_{\text{av}}^{\text{pch}}\) is defined such that for each of the variations, the total heat supplied to the channel with an \(N_{\text{pch}}(\eta)\) is the same as that supplied at a uniform value of \(N_{\text{av}}^{\text{pch}}\).

The linear stability analysis and the determination of the stability boundaries using the D-partition method are discussed in Achard et al. (1985) for the case of a constant \(N_{\text{pch}}\). This allows us to determine if a particular operating point characterised by \(v_{\text{in}}^{\text{av}}\) is stable or unstable. Here a two dimensional parameter space is divided into three regions \(D_0, D_1, D_2\) by two boundaries LSB and MSB. In region \(D_0\) we have a stable operation at the desired operating point, while for parameters in \(D_1\) and \(D_2\) the operating point is unstable. We extend the method for the case of an axially varying heat flux. To ensure a fair comparison between the different cases we ensure that the total heat supplied for the case of an axial variation in heat flux is the same as that supplied with a uniform heat flux at a constant value \(N_{\text{av}}^{\text{pch}}\).

For parameter combinations in \(D_1\) the operating point is in the negative slope region of the steady-state characteristic and is statically unstable. The operating point in regions \(D_0, D_2\) are in the positive slope region. For parameter combinations in \(D_0\), the operating point is stable while in \(D_2\) the operating point is unstable and exhibits
sustained oscillations. Since we are going to focus only on dynamic behaviour we discuss the influence of an axial variation in heat flux on the MSB which separates $D_0$ and $D_2$. In particular we do not depict the Ledinegg stability boundary in the D-partition diagrams which follow.

The effect of a linearly increasing axial heat flux ($\epsilon > 0$) is depicted in Fig. 3. The MSB for the case of a constant $N_{pch}$ is shown by the solid line. The MSB for the linear variation ($\epsilon > 0$) shown by the dotted line is shifted to the right for every $N_{sub}$. This increases the region of stable operation (i.e. $D_0$). This spatial variation can hence be interpreted as having a stabilising effect. For the linearly decreasing axial heat flux ($\epsilon < 0$), the MSB is shifted to the left for every $N_{sub}$. This variation hence has a destabilising effect as it decreases the region of stable operation ($D_0$). Talyerkhan et al. (1981) studied the effect of a linearly varying heat flux using Nyquist plots. They established that the linearly decreasing heat flux distribution has a destabilising effect on the channel confirming the results of Fig. 3.

The exponential variation of heat flux is similar to the linear variation in the sense that it varies monotonically with distance. The effect of the exponential variation for $\epsilon > 0 (\epsilon < 0)$ can hence be expected to be similar to that of the linear variation with $\epsilon > 0 (\epsilon < 0)$. This effect of an exponential variation of heat flux on the MSB is shown in Fig. 4 for both $\epsilon > 0$ and for $\epsilon < 0$. It can be easily argued based on the shifts in the MSB that the qualitative effects of the linear and exponential variations are similar.

The periodic variation in contrast to the other profiles considered is a non-monotonic function ($k > 1$). The effect of a cosine variation on stability is depicted in Fig. 5. For sufficiently large $N_{sub}$ and with $\epsilon < 0$ the cosine variation has a destabilising effect and the MSB is shifted to the left. For low $N_{sub}$ this variation has a stabilising effect and the MSB is shifted to the right. For $\epsilon > 0$ the qualitative effect on stability is reversed as shown in Fig. 5. There have been contradictory reports on the effect of stability of a periodic variation of heat flux in the literature. These reports are based on experimental results. Djikman (1971) found that cosine flux stabilizes the flow. He suggested that it may be due to the decrease in low pressure drop at the exit where the heat flux is lower than the average. Bergles (1976) suggest that cosine flux is destabilizing. Yadigaroglu and Bergles (1972) have reported a destabilising effect of a periodic variation of heat flux. These contradic-
tory reports in the literature can be explained by the shifts in the MSB as shown in Fig. 5. For a fixed $\epsilon$ the effect of a periodic variation of $N_{\text{pch}}$ is determined by the value of $N_{\text{sub}}$. Similarly for a fixed $N_{\text{sub}}$, $\text{sgn}(\epsilon)$ determines whether the periodic variation is stabilising or destabilising as can be seen in Fig. 5.

The stability boundaries obtained can be validated by simulating the system across the boundaries. For an $N_{\text{sub}}, N_{\text{pch}}$ combination in region $D_0$ with $\Delta P_{\text{imp}}$ chosen appropriately to ensure operation at the desired operating point, simulations tend to the steady-state. Increasing $N_{\text{pch}}$ for a fixed $N_{\text{sub}}$, we cross the MSB and reach region $D_2$. Here the operating point for which the D-partition diagram is drawn is unstable and the system exhibits steady, self-sustained oscillations. The numerical algorithm described in the earlier section is used to simulate the dynamic behaviour here. We validate the algorithm first by simulating the system for a uniform heat flux.

The algorithm described was validated by comparing our numerical results with the results already available in the literature for the dynamic behaviour of the system with a constant pressure drop and uniform heat flux (Rizwan-uddin and Dorning, 1988). The simulations were carried out for $v_{\text{in}}^\text{ss} = 1$ with the parameter vector $(k_{\text{in}}, k_{\text{ex}}, \text{Fr}^{-1}, N_{11}, N_{12}, N_{\text{sub}}, N_{\text{pch}}) = (6, 2, 30, 2.8, 5.6, 7.5, 13)$. The corresponding steady-state pressure drop is $\Delta P_{\text{imp}} = 56.84$. This parameter set falls to the right of the MSB in $D_2$. For this set, the steady-state is unstable. A complex-conjugate pair of eigen-values of the characteristic equation is in the right-half plane and we expect DWO. The instantaneous location of the interface was obtained using the method of finite differences. The integrations were carried out using 1400 grid points and a $\Delta \tau$ of 0.005. The evolution of the inlet velocity with time and a phase plane plot of $v_{\text{in}}(\tau)-\xi(\tau)$ is shown in Fig. 6. The inlet velocity varies between $v_{\text{min}}$ (0.42) and $v_{\text{max}}$ (1.57). The frequency of oscillation $\omega_n = 2.86$. These compare favourably with the values of $v_{\text{min}}$ (0.4) and $v_{\text{max}}$ (1.6) (as read from the graph) and $\omega_n = 2.909$ of Rizwan-uddin and Dorning (1990).

The system behaviour when subject to a periodic variation in heat flux was simulated across the MSB at points B, C (Fig. 5). The point B is in

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**Fig. 5.** The effect of a periodic variation of $N_{\text{pch}}$ on MSB (a) $\epsilon \leq 0$, (b) $\epsilon \geq 0$, for parameter set as in Fig. 3.

**Fig. 6.** Periodic inlet velocity variation with time for unforced system. (a) Velocity versus time, (b) velocity-interface position phase plane.
Table 1

| $N_{\text{sub}}$ | $N_{\text{pch}}$ | Region | Scheme 1 | | | Scheme 2 |
|---|---|---|---|---|---|
| 7.5 | 13 | $D_2$ | $v_{\text{max}}$ | 1.70 | 1.57 |
| 6.0 | 11.2 | $D_2$ | $v_{\text{min}}$ | 0.24 | 0.42 |
| 5.0 | 9.9 | $D_0$ | Number of grid points | 6 | 6 |

the stable region of operation ($D_0$) for the periodic variation in heat flux with $\epsilon > 0$, and is in the unstable region of operation for the constant heat flux. Disturbances in the system decay to the steady-state for the former but in the latter they become amplified and the system evolves to a limit cycle. The point $C$ is in the unstable region for both distributions of heat flux. So now the system exhibits oscillations for both distributions.

5.2. Periodic variation of $\Delta P_{\text{imp}}$

The density wave oscillations described occur under conditions of constant imposed pressure drop. They occur due to the non-linear interactions of the system and are not due to any external periodic forcing. The frequency of DWO is the natural frequency $\omega_n$ of the system. In many situations the imposed pressure drop $\Delta P_{\text{imp}}$ may not be constant. This could arise for example when the pressure head generated by the pump varies periodically with time. To simulate the behaviour under conditions of varying $\Delta P_{\text{imp}}$ we impose a sinusoidal variation around the steady-state value $\Delta P_{\text{ss}}$, of the form

$$\Delta P_{\text{imp}}(\tau) = \Delta P_{\text{ss}}(1 + \epsilon \sin \omega_f \tau)$$

where $\omega_f$ is the forcing frequency and $\epsilon$ the forcing amplitude. The numerical scheme discussed earlier can be used to incorporate this effect. The only change is that the velocity at every time instant is determined by satisfying the imposed pressure drop at that instant. The periodic variation of imposed pressure drop introduces a second time scale into the problem. The interaction between the natural frequency and the forcing frequency gives rise to different dynamic behaviours such as subharmonic oscillations, quasi-periodic and chaotic solutions. Our algorithm was able to simulate all these phenomena. We also found the occurrence of different kinds of oscillations called ‘relaxation oscillations’. We discuss all these different dynamic behaviours in Section 5.2.1.

5.2.1. Relaxation oscillations

It is well known that the steady-state characteristics of our system can exhibit multiple steady-states (Quandt, 1961). Consider the case where the parameter set is such that we can have three values for the steady-state inlet velocity for a fixed pressure drop as shown in Fig. 7a. Let $\Delta P_{\text{ss}}$ lie between $\Delta P_{\min}$ and $\Delta P_{\max}$ as indicated in Fig. 7a and $\epsilon$ be such that $\Delta P_{\text{imp}}(\tau)$ varies between a value greater than $\Delta P_{\max}$ to a value lower than $\Delta P_{\min}$. In the limit of low $\omega_f$, the variation of the $\Delta P_{\text{imp}}(\tau)$ is very slow. The system can be expected to be at a pseudo-steady state at every instant of time. The slow variation of $\Delta P_{\text{imp}}(\tau)$ consequently forces the system to trace out the steady-state characteristics.

We simulated these relaxation oscillations for the parameter vector $[6, 2, 30, 2.8, 5.6, 7.5, 20]$ with $\omega_f$, $\epsilon$ and $v_{\text{ss}}$ chosen as 0.1, 0.017, and 1.78, respectively (Fig. 7b). Let the system be operating at the point $A$ when the imposed pressure drop is increasing with time. The system moves along the path AB. An increase of $\Delta P_{\text{imp}}(\tau)$ beyond $\Delta P_{\max}$ (the value at point $B$) forces the system to depart from the branch AB and relax to the steady-state at $C$. The transition from $B$ to $C$ is very fast (essentially instantaneous). The steady-state at $C$ is the only feasible state for $\Delta P_{\text{imp}}(\tau) > \Delta P_{\max}$. A decrease in $\Delta P_{\text{imp}}(\tau)$ now forces the system to follow the path $CD$ (the high velocity branch). In
particular the decrease in $\Delta P_{\text{imp}}$ at C, does not force the system from operating point C back to B. The system exhibits hysteresis. As $\Delta P_{\text{imp}}$ decreases to lower than $\Delta P_{\text{min}}$, the system shows another rapid excursion from point D to point E. An increase in $\Delta P_{\text{imp}}$ now makes the system follow the path EAB as shown.

This kind of an oscillation is called a relaxation oscillation. It is caused by the presence of two widely varying time scales in the system. This is reflected in the slow variation along the branch from A to B and C to D and the fast excursions from B to C and D to E. The high frequency oscillations which occur in the time series for low $v_{\text{in}}$ (Fig. 7b) arise because the unforced system exhibits DWO along EB for the chosen set of parameter values.

The relaxation oscillations appear to be very similar in form to the pressure drop oscillations (PDO) observed experimentally by Padki et al. (1992). It must be remembered that the PDO investigated occur when there is a surge tank included upstream of the boiling channel. Here the surge tank pressure is effected by the velocity exiting from it and the velocity entering it. The inlet velocity to the channel is in its turn effected by the surge tank pressure. This coupling or feedback effect of the velocity in the channel on the pressure drop across the channel does not exist in the oscillations of Fig. 7b where we have assumed the pressure drop to be imposed from external considerations.

5.2.2. Sub-harmonic and chaotic oscillations

We simulated our system behaviour for $\omega_f = 5.79$ and $\epsilon = 0.005$. Now the forcing period is one-half of the natural period of oscillations. The system now oscillates (after the initial transients have decayed) at a period equal to twice the forcing period. This solution is called a sub-harmonic solution or more specifically a 2P solution. Assuming that only three modes are dominant in this infinite-dimensional system we depict this attractor in a three dimensional phase space in Fig.

Fig. 7. Relaxation oscillations when multiple steady-states co-exist (a) steady-state characteristics, (b) inlet-velocity variation.

Fig. 8. Subharmonic oscillations (a) phase-space representation, (b) stroboscopic map of a 2-P solution.
8. The three variables chosen for the representation are $v_n(t)$, $\lambda(t)$, $\Delta P_{\text{imp}}(t)$ which are assumed to be independent.

The attractor in Fig. 8a is a closed curve, signifying that it is periodic. To determine the period of the solution we generated its stroboscopic map. The signal, i.e. velocity time series is strobed at a frequency equal to the forcing frequency. The values of the variables $v_n(t)$, $\lambda(t)$ on the attractor which are separated by a forcing period are plotted. This corresponds to determining the intersection of the trajectory with a plane of constant $\Delta P_{\text{imp}}$ such that at the intersection $(d\Delta P_{\text{imp}}/dt) > 0$ or $(d\Delta P_{\text{imp}}/dt) < 0$. The stroboscopic map is an elegant way to generate the Poincare map in forced systems. For the subharmonic 2P solution the stroboscopic map yields two points (Fig. 8b). The spread of points in the map occurs due to (i) the tolerance in the Newton–Raphson step of the numerical algorithm and (ii) the approximation of the strobing period. We increased $\epsilon$ to see if a period-doubling cascade exists for a fixed set of all parameters. We were unable to detect any period-doubling bifurcations in our simulations.

For low values of $\epsilon$ the two frequencies $\omega_f$, $\omega_h$ do not get inter-locked. The resulting state is quasi-periodic. The three-dimensional phase-space portrait of such an attractor is shown in Fig. 9a for $\omega_f = 6.653$. The stroboscopic map of such an attractor yields a closed curve (Fig. 9b). The trajectory now is characterised by the presence of two frequencies whose ratio is an irrational number. Such a trajectory can be viewed as lying on a toroidal surface. This behaviour is typical of forced systems. Our extensive simulations revealed the presence of quasi-periodic solutions for a large combination of $\omega_f$ and $\epsilon$. The sub-harmonic solutions occur over a very narrow range of parameter space. Detecting these require specifying parameters with a very high accuracy for simulations.

For $N_{\text{pch}}$ at 11.8 with $v_{\text{in}}^{ss} = 1$ the autonomous or the unforced system has a stable steady state with a channel pressure drop of $\Delta P_{\text{ss}} = 52.66$. The forced system with $\omega_f = 11.58$, and $\epsilon = 0.07$, evolves to a periodic state with a period equal to the forcing period. As $N_{\text{pcf}}$ is increased to 12.4, the channel pressure drop $\Delta P_{\text{ss}}$ is changed to 54.7 to allow for a feasible steady-operation at $v_{\text{in}}^{ss} = 1$. The autonomous or unforced system exhibits DWO for this combination of parameters and the steady state at $v_{\text{in}}^{ss} = 1$ is unstable. The forced system with the above set of forcing parameters now evolves to a quasi-periodic state.

As we increase $N_{\text{pcf}}$ further to 13, the attractor becomes chaotic. The corresponding pressure drop $\Delta P_{\text{ss}}$ is changed to 56.84 to maintain $v_{\text{in}}^{ss} = 1$. This chaotic attractor in the three-dimensional phase space and its corresponding stroboscopic map are shown in Fig. 10(a,b).

5.2.3. Analysis of behaviour using the time series data of a single variable

A dynamic system is characterised by the variation of its dependent variables with time. In a theoretical study when the model is assumed, dynamic simulations predict the evolution of all

Fig. 9. (a) Phase space representation of a quasi-periodic solution, (b) stroboscopic map.
the dependent variables. In an experimental study it is often difficult to measure all variables. The phase space representation in Figs. 8 and 9 needs information about \( \lambda(t) \). In this system it is difficult to measure \( \lambda(t) \) experimentally. The representation of the system in terms of phase portraits now loses its significance as the different dependent variables cannot be measured (at least not easily). Further, the processes occurring in the system are not always well understood and it is then difficult to accurately model the system. It is hence necessary to describe the system behaviour based on measurements of a single variable. The technique of time delay coordinate embedding can be used to characterise such systems using the data on measurements of a single-variable. The single-variable we consider as measurable is the velocity entering the evaporator. The numerical algorithm described for the dynamic simulation is used to generate the dynamic behaviour of the system. The velocity time series generated is interpreted as the results of an experiment now. The attractor is reconstructed using the time delay embedding method described in Grassberger and Procacia (1983), Leibert and Schuster (1991). We now discuss how the method can be used to characterise the dynamic behaviour of the system using velocity time series data. It also enables us to determine the minimum number of variables required to describe the system completely. The reconstructed attractor is obtained from the following steps

1. The time series \( v_{in}(t) \) is generated by simulations. The time interval of integration is 0.005 (in dimensionless time units). This time series is viewed as experimental data obtained by sampling the inlet velocity at the above time period. The system can be characterised quantitatively if we can determine the embedding dimension \( d \) and delay time \( T_d \). Optimum values of these variables is essential for the successful computation of the attractor dimension \( \gamma \).

2. The time delay \( T_d \) is estimated in terms of the sampling time using the method of Leibert and Schuster (1991). A sample of 5000 points (after the initial transients have died down) was used to determine the delay time. The first 5000 points were omitted to discard transient effects. The correlation integral for embedding dimension \( d = 3, 4 \) are shown in Fig. 11. The correlation integral is computed using

\[
C(r) = \sum_i \sum_j \frac{1}{N^2} \text{no. of points} \| v(i) - v(j) \| \leq r
\]

The two curves exhibit a first minimum at the point where the ratio delay time/sampling time is 4.0 for \( r = 0.005 \). So the delay time was chosen as 0.02. Since the result was invariant with \( d \) varying from 3 to 4 we choose the embedding dimension to be 3. This implies that if we were to construct a phenomenological model to describe the oscillatory behaviour it would require a minimum of three independent variables.
3. A three-dimensional time series is generated using the original time series. The different elements of the time series are $v(t)$, $v(t + T_d)$, $v(t + 2T_d)$. It is generated by going down the original single time series. This three dimensional time series contains the same information as three independent variables which would characterise the system behaviour.

4. A point in the reconstructed attractor is specified as $\bar{v}(t)$, $[v(t), v(t + T_d), v(t + 2T_d)]$.

5. The correlation dimension of the attractor $\gamma$ is defined as $C(r) = Ar^\gamma$. The time series data of velocity for a chaotic attractor was simulated using $N_{pch} = 13$, $\omega_i = 11.58$ and $\epsilon = 0.07$.

The reconstructed attractor in phase-plane using $T_d$ of 0.02 from the above time series is shown in Fig. 12a. The dimension of the attractor is obtained from a log–log plot of $C(r)$ versus $r$. To obtain a good estimate of $\gamma$, the range of $r$ for which this plot should be drawn is such that it covers the entire size of the attractor. The attractor dimension using the time series $v_{in}(t) - \lambda(t) - \Delta P_{imp}(t)$ (since these are not dependent on each other linearly) is shown in Fig. 12b. Here we compare the attractor dimension with the correlation dimension as generated by the reconstructed attractor on a log–log plot. The two curves are almost parallel, indicating that they have the same slope. The correlation dimension of the attractor in the $v - \lambda - \Delta P$ space is 2.4 and that of the reconstructed attractor is 2.45.

![Figure 11](image1.png)

**Fig. 11.** Estimation of a delay-time from velocity time series using correlation integrals.

![Figure 12](image2.png)

**Fig. 12.** (a) Phase-space representation of the reconstructed chaotic attractor. (b) Comparison of attractor dimensions of reconstructed attractor with original attractor.

In Fig. 13 we have depicted a periodic attractor using the time delay coordinate embedding method. The reconstructed attractor using a delay time of 0.02 corresponds to the periodic attractor shown in Fig. 8.

The delay-coordinate embedding method is a useful method when only a single-variable can be measured. We have shown how it can be used to successfully determine the minimum number of modes which are important in a dynamic system. The number of modes is the embedding dimension which for our system was three. This justifies
using three variables (which we chose as \(v(t) - \lambda(t) - \Delta P(t)\)) in Fig. 9a to represent system behaviour.

Our dynamic system is an infinite dimensional system being governed by a system of partial differential equations. It would hence theoretically require an infinite number of variables to characterise its behaviour. However the actual behaviour will be determined only by a few dominant modes of interaction. The number of these dominant modes is the embedding dimension. The embedding dimension can also be used to generate a simplified phenomenological model of a complex system.

6. Conclusions

In this work we have presented the results of a numerical investigation of a two phase flow system occurring in an evaporating channel. The technique is a shooting method technique based on an implicit finite difference. It is important to use backward finite differences for spatial derivatives, as the information in our system flows from bottom to top. Our numerical technique is also sufficiently general to incorporate the variation of heat flux with axial position. The method of characteristics used by earlier workers is applicable only when the heat flux parameter \(N_{pch}\) is a constant.

We have investigated the effects on the MSB for three different axial heat flux variations. The effect on stability of the linear and exponential variations is independent of \(N_{sub}\) but for the periodic variation depends on \(N_{sub}\).

We have simulated complex dynamic behaviour like relaxation oscillations, quasi-periodic behaviour and chaotic solutions of our system by incorporating the effect of a periodic variation in the imposed pressure drop. We have also shown the applicability of the delay-coordinate embedding technique to characterise our system using experimental results which provide the time series data of inlet velocity. The accuracy of this technique was established by comparing it with the attractor features of the original infinite-dimensional system.

Appendix A. Nomenclature

\[
\begin{align*}
 v_g & \quad \text{vapour velocity} \\
 v_l & \quad \text{liquid velocity} \\
 v_{in} & \quad \text{inlet velocity} \\
 j & \quad \text{volumetric flux} \\
 \eta & \quad \text{vertical distance} \\
 \lambda & \quad \text{position of interface} \\
 \tau & \quad \text{time} \\
 \rho_M & \quad \text{mixture density} \\
 P & \quad \text{pressure} \\
 Fr & \quad \text{Froude number} \\
 N_r & \quad \text{ratio of densities} \\
 N_{f1} & \quad \text{single phase friction number} \\
 N_{f2} & \quad \text{two phase friction number}
\end{align*}
\]

Fig. 13. Reconstructed periodic attractor corresponding to attractor of Fig. 8.
\[ N_{\text{sub}} \frac{\Delta h_{\text{fo}}^* (\rho_i - \rho_g)}{\Delta h_{\text{fg}}^* \rho_g} \text{, subcooling number} \]

\[ N_{\text{pch}} \frac{g_{\text{lo}} c_{\text{fg}}^* \frac{L^*}{\rho_f^*} \Delta \rho^*}{A^* \Delta h_{\text{fg}}^* \rho_f^* v_{\text{bo}}^*} \text{, phase change number} \]

References


